The Coleman-Shapley index: Being decisive within the coalition of the interested

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Abstract

The Coleman power of a collectivity to act (CPCA) is a popular statistic that reflects the ability of a committee to pass a proposal. Applying the Shapley value to that measure, we derive a new power index—the Coleman-Shapley index (CSI)—indicating each voter's contribution to the CPCA. The CSI is characterized by four axioms: anonymity, the null voter property, the transfer property, and a property stipulating that the sum of the voters' power equals the CPCA. Similar to the Shapley-Shubik index (SSI) and the Penrose-Banzhaf index (PBI), our new index is the expectation of being a pivotal voter. Here, the coalitional formation model underlying the CPCA and the PBI is combined with the ordering approach underlying the SSI. In contrast to the SSI, voters are ordered not according to their agreement with a potential bill, but according to their vested interest in it. Among the most interested voters, power is then measured in a way similar to the PBI. Although we advocate the CSI over the PBI so as to capture a voter's influence on whether a proposal passes, our index gives new meaning to the PBI. The CSI is a decomposer of the PBI, splitting the PBI into a voter's power as such and the voter's impact on the power of the other voters by threatening to block any proposal. We apply our index to the EU Council and the UN Security Council.

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Contents

1 Introduction

Coleman's (1971) power of a collectivity to act (CPCA) is a popular measure for the decisiveness of a committee; i.e., it reflects the ease with which the individual attitudes toward a proposal can be translated into whether this proposal will actually pass. It is accepted and applied across disciplines (Felsenthal and Machover 2001, 2004, Leech 2002b,a, Ibarzabal and Laruelle 2018). It is a simple and transparent committee-level statistic that counts the share of winning coalitions among all possible coalitions that may form in a committee. The Penrose-Banzhaf index (PBI) is an individual-level statistic that measures each voter's power in a committee (Penrose 1946, Banzhaf 1965). It is the decline in the CPCA of a committee caused by a voter's abstention or a voter's rejection of proposals. As a consequence, the sum of the PBI of all voters provides an expression that itself is hard to interpret and that is not applicable to comparisons across committees. Shapley and Shubik (1954) propose another index measuring the power of an individual in a committee. This Shapley-Shubik index (SSI) differs from the PBI in that it always sums to one. Conceptually, the SSI indicates the influence of a committee member on *which* bill is going to pass; i.e., it is assumed that something will be decided and the question is in whose favor. The PBI, in contrast, refers to the influence on *whether* a bill is going to pass (cf. Dubey and Shapley 1979).

In this paper, we propose a new power index—henceforth called the Coleman-Shapley index (CSI)—which resembles the PBI as it follows from the CPCA and which resembles the SSI in that the power of all voters sums to a meaningful term. Note that if either a coalition or its complement can be decisive but not both, i.e., if the committee is non-contradictory, then at most half the coalitions can be winning and the CPCA lies between 0 and 0.5. As one usually is interested in non-contradictory committees, we consider the normalization of the CPCA, i.e., twice the CPCA, which we denote by 2CPCA. It is exactly the latter entity that is distributed across the voters by the CSI:

$$\begin{aligned}&\text{Sum of CSI over all committee members}\\ &\qquad\qquad = 2 \times \text{Coleman power of the collectivity to act}\end{aligned}$$

That property, together with the standard axioms of transfer, symmetry, and null voter, is characteristic of the CSI. Having an index that sums up to a interpretable term not only is of practical advantage (e.g., for cross-committee comparison) but also supports the soundness of the concept, in particular, given a clear understanding of the overall power of a committee in terms of 2CPCA. We therefore advocate the CSI over the PBI as a measure of a voter's influence on whether a proposal passes—even though the CSI gives new meaning to the PBI.

The CSI is a decomposition of the PBI, a concept studied by Casajus and Huettner (2017). Consequently, the CSI splits PBI into a direct part (reflecting a voter's influence on whether a proposal passes) and an indirect part (reflecting the impact of her threat no longer to support any proposal on the other voters' direct power):

$$\begin{aligned}&\text{A voter's power according to PBI}\\ &\quad = \text{her power according to CSI}\\ &\qquad + \text{what the others gain or lose according to CSI}\\ &\qquad\quad \text{when she no longer supports any proposal}\end{aligned}$$

In that sense, the PBI captures not only a voter's power, but also her impact on the power of the other voters associated with her threats. This induces a "double count" if the PBI is summed across all voters and explains the problematic behavior of the PBI.

Both the SSI and the PBI can be obtained as the expectation of being a pivotal voter. The CSI also emerges from such a model. To this end, we assume that voters may be more or less interested in the topic to be decided. Let us rank the voters according to their interest, beginning with the voter who will most definitely show up first when the topic is on the agenda and ending with the voter who is least interested. Assume that all orderings regarding interest are equally likely. In expectation, a voter thus faces other voters of the committee who are more interested than she is and who already engage with the topic. Within those most interested voters, she may now be pivotal in the sense that her joining the subcoalition comprising those voters in favor of the proposal and more interested than she is makes this subcoalition a winning coalition. Given that being interested in a topic and being in favor of a proposal are independent behind the veil of ignorance, the CSI emerges as the expectation of being pivotal:

$$\text{CSI} = \text{expectation of being pivotal among the most interested}$$

The three equations above constitute the cuneate contributions of our paper: we suggest a power index that (i) distributes a meaningful measure of committee decisiveness—the Coleman power of a collectivity to act—among the voters, (ii) splits the Penrose-Banzhaf index into a constructive power component and a veto power component, and (iii) measures the expected value of being a pivotal voter among the voters most interested in the topic at stake. Consequently, our index facilitates comparison of (individual) power across committees, provides a better understanding of the Penrose-Banzhaf index, and integrates the idea of voters having priorities on topics in which they may engage.

The remainder of this paper is organized as follows. Section 2 surveys related literature. Section 3 introduces and characterizes the CSI. The relationship between the CSI, the Shapley value, and the PBI is studied in Section 4. In Section 5, we consider examples. Some remarks conclude the paper. Online supplementary materials support the computation of the index.

2 Related literature

For a survey of other power indices, we refer to Bertini et al. (2013). A rich literature compares the PBI and the SSI. Many contributions are of axiomatic character. Employing standard axioms—transfer, symmetry and null player – the difference in both indices boils down to the sum of the power held by the voters (Dubey and Shapley (1979), Laruelle and Valenciano (2001)). Einy and Haimanko (2011) characterize the SSI by weakening the assumption that power sums to one. Instead, they require that whenever a voter gains power if the voting rule of a committee changes, another voter needs to lose power. Lehrer (1988), Casajus (2012, 2014), and Haimanko (2017) employ equivalences of payoffs by amalgamating two voters into a singleton. While such amalgamation clearly is of mathematical interest, it is rather counterintuitive for an index measuring the impact of voters on whether a proposal passes (joining forces typically increases that impact). However, in light of the decomposition of the PBI by the CSI, that property gains plausibility.

Beyond the axiomatic approach, Felsenthal and Machover (1998) stress paradoxes resulting from illicit (mis)understandings of indices. They further distinguish two notions of voting power: I-power and P-power. Whereas I-power represents the degree to which a player determines whether a coalition is winning or losing, P-power indicates a player's expected share in a fixed prize gained by a winning coalition containing her. We present a cooperative game that assigns to every coalition an I-power score, and we use the Shapley value to divide this I-power among the voters. Therefore, the CSI can be considered an I-power index. Wiese (2009) extends the scope of notions of power by discussing the idea of power over others as power that emerges from threats to no longer support any proposal and connects cooperative game theory to the sociology literature. We connect to this notion of power over others when decomposing the PBI into a direct power component and a voter's impact from threats, where the latter can be understood as power over others.

Some empirical analyses favor the PBI, e.g., Leech (1988) and Renneboog and Trojanowski (2011), while others rest on the SSI, e.g., Köke and Renneboog (2005), Basu et al. (2016), and Bena and Xu (2017). Finally, there are non-cooperative game theoretical studies of voting systems that may support one or the other power index. To this end, we refer to Kurz et al. (2017b,a), who derive the SSI in the context of a two-tier voting procedure.

Power indices are applied for various purposes and are often extended to serve a particular purpose. A typical extension manipulates the possibility of particular coalitions forming, e.g., based on judgement about the preferences of the committee members. Amaral and Tsay (2009) use such a modification of the PBI to measure influence in an outsourcing game. Owen (1977) assumes prior unions of voters and imposes that a union can only vote for a proposal with all its members supporting. This invokes an intermediate committee with unions as voters, which is used to measure power—first of the unions and second of their members. Karos and Peters (2015) study mutual control structures, i.e., systems of committees where a committee might also take the role of a voter in another committee (e.g., the shareholders of a company are the voters in the general meeting of this company which in turn might act as a shareholder of another company; cf. Gorton and Schmid (2000)). We remark that these extensions are also possible for the CSI.

Efficient computation of power indices is studied by, among others, Leech (2003), who suggests an approximative algorithm based on the multilinear extension introduced by Owen (1972). Kurz (2016) highlights the efficiency of dynamic programming to count the number of coalitions of a particular size containing a particular player for the computation of power indices in weighted voting games. We provide the means to pursue computation of the CSI along those lines in the appendix. A survey of various algorithms is provided by Matsui and Matsui (2000).

3 The Coleman-Shapley index

Let $N = \{1, \ldots, n\}$ contain the members of an assembly. The voting rules are reflected by a collection $\mathcal{W}$ of subsets of N that satisfies the following properties:[1]

(i) $\emptyset \notin \mathcal{W}$.

(ii) If $S \subseteq T \subseteq N$ and $S \in \mathcal{W}$, then $T \in \mathcal{W}$.

Members $i \in N$ are called voters. Coalitions $S \in \mathcal{W}$ are called winning coalitions. The set of all assemblies is denoted by $\mathbb{W}$. Unless specified otherwise, we further assume that the rules are non-contradictory; i.e., a coalition and its complement cannot both be winning:

(iii) If $S \in \mathcal{W}$, then $N \setminus S \notin \mathcal{W}$.

Coleman (1971, 1990) introduces a measure for the **Power of a collectivity to act** (**CPCA**) that is given by[2]

$$\mathrm{CPCA}(\mathcal{W}) = \frac{1}{2^n} \sum_{S \subseteq N} [S \in \mathcal{W}], \tag{1}$$

where the square brackets are understood to be logical operators,

$$[\text{statement}] = \begin{cases} 0, & \text{statement is false,} \\ 1, & \text{statement is true.} \end{cases}$$

[1]We do without the assumption $N \in \mathcal{W}$ frequently made in the literature. This extends the domain of our axioms. However, our results would also hold true if the axioms were restricted to the smaller domain.

[2]Working on a fixed set N, we omit it in the following and, e.g., write $\mathrm{CPCA}(\mathcal{W})$ instead of $\mathrm{CPCA}(N, \mathcal{W})$.

In other words, the CPCA counts the number of winning coalitions and divides it by the number of all coalitions (including the empty coalition). One interpretation is as follows. With a probability of $1/2$, any of the voters is in favor of a given proposal or against it. This implies that the probability of coalition $S \subseteq N$ being in favor of the proposal while the coalition $N \setminus S$ is against it is $1/2^n$. Thus, the Coleman power of a collectivity to act equals the chance that a proposal meets a winning coalition and passes—followed by some corresponding action; hence the name. For non-contradictory games, it is reasonable to normalize this measure since at maximum only half the coalitions can be winning. We will use **2CPCA** to denote this normalization,

$$\text{2CPCA}(\mathcal{W}) = \frac{1}{2^{n-1}} \sum_{S \subseteq N} [S \in \mathcal{W}] .$$

While the power indices of the collectivity to act refer to the whole assembly N, the Penrose-Banzhaf index measures the voting power of each individual voter $i \in N$. It is given as the probability of being a pivotal voter. Voter i is a pivot in coalition $T \subseteq N,\ i \in T$ if leaving this coalition turns it from a winning coalition into a losing coalition, i.e., $\{i\} \in T \in \mathcal{W}$ and $T \setminus \{i\} \notin \mathcal{W}$. Suppose that every other voter is in favor of (or against) a proposal with probability $1/2$. Then, the probability that exactly the voters of coalition $T \setminus \{i\} \subseteq N \setminus \{i\}$ are in favor of the proposal is 2^{n-1}. The **Penrose-Banzhaf index** (**PBI**) equals the expectation of being a pivotal voter,

$$\text{PBI}_i(\mathcal{W}) = \frac{1}{2^{n-1}} \sum_{T \subseteq N : i \in T} [T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}]$$

for all $\mathcal{W} \in \mathbb{W}$ and $i \in N$, where $[T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}]$ takes the value 1 if i is pivotal in T and the value 0 if i is not pivot in T.

The Shapley-Shubik index (SSI) is based on a different notion of when being pivotal matters, with the consequence that the probability distribution over the set of coalitions that are in favor of the proposal and can be joined by voter i is different. The probability model behind the SSI supposes that the voters have different degrees of support for a bill that is to be decided on. Ordering the voters beginning with the most supportive voter and ending with the most dismissive opponent, reveals a decisive voter along the line who makes the bill pass (or fail). It appears that this voter is targeted most by the others who are trying to convince her into their camp. Consequently, this voter has crucial influence on the details of the bill. Assuming that all orderings are a priori equally probable, then the SSI is the probability of being the decisive voter.[3] Formally, a rank order for N is a bijection $\sigma : N \to \{1, \ldots, n\}$; R denotes the set of all rank orders. For N, $\sigma \in R$, and $i \in N$, we denote the coalition of voter i together with the voters who appear before i in σ by $B_i(\sigma) := \{j \in N \mid \sigma(j) \leq \sigma(i)\}$. The **Shapley-Shubik index** (**SSI**) is given by

$$\mathrm{SSI}_i(\mathcal{W}) = \frac{1}{n!} \sum_{\sigma \in R} \left[B_i(\sigma) \in \mathcal{W} \text{ and } (B_i(\sigma) \setminus \{i\}) \notin \mathcal{W} \right].$$

The Coleman-Shapley index (CSI) captures power in terms of the influence on both whether a bill can be passed (similar to PBI) and which coalition is essentially designing the bill (similar to SSI). Suppose that the voters might have varying interests in investing time and resources on a topic and that this determines whether they are present in the room when a bill is worked on. This invokes an ordering of the voters according to their enthusiasm on a topic, beginning with the voter who is most interested and eager to work on a bill, and ending with the voter who is least concerned about the topic. We assume that all such orderings are equally probable. Within this coalition of the most interested voters $B_i(\sigma)$, it is now crucial to make a difference in whether a bill can be passed. Assuming that preferences are for or against with equal probabilities (as for the CPCA or the PBI), this is measured by $1/2^{|B_i(\sigma)|-1} \times |\{S \subseteq B_i(\sigma) \mid i \in S \in \mathcal{W} \text{ and } S \setminus \{i\} \notin \mathcal{W}\}|$. The *expectation of being pivotal in a coalition of most interested* voters is then equal to the **Coleman-Shapley index (CSI)**, given by

$$\mathrm{CSI}_i(\mathcal{W}) = \frac{1}{n!} \sum_{\sigma \in R} \frac{1}{2^{|B_i(\sigma)|-1}} \sum_{T \subseteq B_i(\sigma) : i \in T} \left[T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W} \right].$$

In what follows, we discuss this new index. We begin with a characterization. The property that distinguishes the CSI from SSI and PBI is the sum across the voters.

2CPCA-Efficiency, 2CE. For all $\mathcal{W} \in \mathbb{W}$, we have

$$\sum_{i \in N} \varphi_i(\mathcal{W}) = 2\mathrm{CPCA}(\mathcal{W}).$$

The motivation for this property arises from the idea that the overall power of a committee is measured by the model of Coleman. However, $\mathrm{CPCA}(\mathcal{W}) = 1/2$ if $\mathcal{W}$ is non-contradictory and has the maximal number of winning coalitions, e.g., if there is a dictator. We therefore suggest the normalized CPCA,

[3] Mann and Shapley (1964), Felsenthal and Machover (1996), Kurz and Napel (2018) and Bernardi and Freixas (2018) generalize the this definition/interpretation of the Shapley-Shubik index by considering more or less general probability distributions on roll call votes, which consist of a rank ordering of the voters and information on their voting behavior, yea or nay.

i.e., twice the CPCA. The following properties are standard.

Anonymity, A. For all permutations σ of N and all $\mathcal{W} \in \mathbb{W}$, we have $\varphi_{\sigma(i)}(\mathcal{W}) = \varphi_i\left(\sigma^{-1}(\mathcal{W})\right)$.

Null voter, N. For all $\mathcal{W} \in \mathbb{W}$ such that $i \in N$ is a null voter in $\mathcal{W}$ (i.e., $S \in \mathcal{W}$ implies $(S \setminus \{i\}) \in \mathcal{W}$), we have $\varphi_i(v) = 0$.

Transfer, T. For all $\mathcal{V}, \mathcal{W} \in \mathbb{W}$, and $i \in N$, we have $\varphi_i(\mathcal{V} \cap \mathcal{W}) + \varphi_i(\mathcal{V} \cup \mathcal{W}) = \varphi_i(\mathcal{V}) + \varphi_i(\mathcal{W})$.

Anonymity just requires that the index does not depend on the labeling of the voters. The null voter property normalizes the index such that a voter without influence receives zero and a dictator receives 1. The transfer property is less innocuous. Laruelle and Valenciano (2001) offer an equivalent version. Consider a coalition S that is minimal winning in both $\mathcal{V}$ and $\mathcal{W}$, i.e., $S \in \mathcal{V}$ and $S \in \mathcal{W}$, but for every strict subset $T \subsetneq S$ we have $T \notin \mathcal{V}$ and $T \notin \mathcal{W}$. Then, the transfer property is tantamount to the requirement that $\varphi_i(\mathcal{V}) - \varphi_i(\mathcal{W}) = \varphi_i(\mathcal{V} \setminus \{S\}) - \varphi_i(\mathcal{W} \setminus \{S\})$ for every voter $i \in N$; i.e., the difference of the power of a voter in two committees is preserved if we make the same minimal winning coalition in both committees losing. If we keep in mind that a committee is specified by its collection of minimal winning coalitions, the transfer property implies that differences in the power of a voter in two committees originate from differences in the committees, i.e., their different minimal winning sets. We can now present a characterization of the CSI.

Theorem 1 *The CSI is the unique power index that satisfies 2CPCA-Efficiency* ***(2CE)****, Anonymity (***A***), Null voter* ***(N)****, and Transfer* ***(T)****.*

Proof We leave it to the reader to verify that the CSI satisfies **2CE**, **A**, **N**, and **T**. To show uniqueness, assume that φ satisfies the four properties. For $T \subseteq N$, $T \neq \emptyset$, let $\mathcal{W}_T = \{S \mid T \subseteq S\}$ denote the unanimity committee of T. By Lemma 2.3 in Einy (1987), φ is already determined by the numbers assigned to $\mathcal{W}_T$. It remains to be shown that these are unique given the properties of φ. Indeed, **2CE** of φ implies $\sum_{i \in N} \varphi_i(\mathcal{W}_T) = 2^{n-t}/2^{n-1}$. Now, **N** implies $\varphi_i(\mathcal{W}_T) = 0$ for $i \notin T$ and consequently $\sum_{i \in T} \varphi_i(\mathcal{W}_T) = 2^{1-t}$. Finally, **A** implies $\varphi_i(\mathcal{W}_T) = 2^{1-t}/t$ for $i \in T$. □

The present characterization allows us to nail down the difference between CSI, SSI, and PBI to just one property. Replacing 2CPCA-Efficiency in the upper theorem with efficiency, i.e., the requirement that the voters' power sums up to one, $\sum_{i \in N} \varphi_i(\mathcal{W}) = 1$, gives a characterization of the SSI. Replacing 2CPCA-Efficiency in the upper theorem with PBI-efficiency, i.e., the requirement that the voters' power sums up to the PBI, $\sum_{i \in N} \varphi_i(\mathcal{W}) = \sum_{i \in N} \mathrm{PBI}_i(\mathcal{W})$, gives a characterization of the PBI.

4 The CSI, the Shapley value, and the PBI

To gain a better understanding of the CSI, it is helpful to study a particular auxiliary TU game. A TU game with player set N is a function v that assigns to every coalition $S \subseteq N$ a number $v(S)$ with the interpretation that $v(S)$ is the worth that can be created by this coalition. By convention, $v(\emptyset) = 0$. Note that a so-called simple TU game can be understood as a representation of a committee. A game is simple if $v(S) \in \{0,1\}$ for all $S \subseteq N$, $v(S) \leq v(T)$ whenever $S \subseteq T$. Coalition S is then said to be winning if $v(S) = 1$ and losing if $v(S) = 0$. Next, we want to investigate a different TU game associated with a committee, a game that is not to be confused with the representation of a committee.

For every $\mathcal{W}$, define the (non-simple) TU game $v^{\mathcal{W}}$ as

$$v^{\mathcal{W}}(S) = \frac{1}{2^{s-1}} \sum_{T \subseteq S} [T \in \mathcal{W}]. \tag{2}$$

The game $v^{\mathcal{W}}$ assigns to every coalition S its share of possible winning subcoalitions. Analogously to the $2\mathrm{CPCA}$, $v^{\mathcal{W}}(S)$ measures *the power of coalition S to act* (if the other voters $N \setminus S$ reject any proposal). Assuming that every voter is equally likely to be in favor of or against a proposal, $v^{\mathcal{W}}(S)/2$ is the probability that a proposal is supported by a winning subcoalition $T \subseteq S$ in $\mathcal{W}$. Consider, for example, the committee with $N = \{1,2,3\}$ with simple majority rule. One of the four subcoalitions of $\{1,2\}$ is winning such that $v^{\mathcal{W}}(\{1,2\}) = 0.5$ reflects the power of $\{1,2\}$ to act. Moreover, $v^{\mathcal{W}}(\{1\}) = 0$ and $v^{\mathcal{W}}(\{1,2,3\}) = 1 = 2\mathrm{CPCA}(\mathcal{W})$.

Next we show that using the auxiliary TU game $v^{\mathcal{W}}$, we can understand the PBI as a voter's contribution to the power of the other voters to act. In other words, PBI_i quantifies the reduction of the power of the collectivity to act if voter i refuses to support any proposal.

Proposition 1 *For every $\mathcal{W}$ and all $i \in \mathcal{W}$, we have*

$$\mathrm{PBI}_i(\mathcal{W}) = v^{\mathcal{W}}(N) - v^{\mathcal{W}}(N \setminus \{i\}),$$

where $v^{\mathcal{W}}$ is given in (2).

Proof Straightforward computation yields

$$v^{\mathcal{W}}(N) - v^{\mathcal{W}}(N \setminus i) = \frac{1}{2^{n-1}} \sum_{T \subseteq N} [T \in \mathcal{W}] - \frac{1}{2^{n-2}} \sum_{T \subseteq N : i \in T} [T \setminus \{i\} \in \mathcal{W}]$$

$$= \frac{1}{2^{n-1}} \sum_{T \subseteq N : i \in T} [T \in \mathcal{W}] - \frac{1}{2^{n-1}} \sum_{T \subseteq N : i \in T} [T \setminus \{i\} \in \mathcal{W}]$$

$$= \frac{1}{2^{n-1}} \sum_{T \subseteq N : i \in T} [T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}],$$

which completes the proof. □

There are two basic arguments against measuring an individual's performance in a TU game by looking at the contribution to the others. First, doing so and summing up over all individuals yields a problematic term. In particular, the right-hand side of the expression

$$\sum_i \text{PBI}_i(\mathcal{W}) = \sum_i \left(v^{\mathcal{W}}(N) - v^{\mathcal{W}}(N \setminus \{i\})\right)$$

is difficult to interpret. Second—and conceptually more important—the last marginal contribution $v^{\mathcal{W}}(N) - v^{\mathcal{W}}(N \setminus i)$ cannot simply be attributed to an individual but also requires participation of other voters (contrast this with $v^{\mathcal{W}}(i) - v^{\mathcal{W}}(\emptyset) = [i \text{ is a dictator}]$, which can be seen as an indicator of voter i's individual performance). Concretely, claiming that voter i's share of the 2CPCA is reflected by PBI_i is flawed because it gives some credit to i for the other voters providing their power to act.

Consequently, we are interested in a concept that disentangles this interrelationship inherent in a cooperatively generated power to act. The most prominent solution to this end is the Shapley value (Shapley 1953), given by

$$\text{Sh}_i(v) = \frac{1}{n!} \sum_{\sigma \in R} \left(v(B_i(\sigma)) - v(B_i(\sigma) \setminus \{i\})\right).$$

As is shown in the next result, it turns out that this yields precisely the CSI.

Theorem 2 *For every $\mathcal{W}$ and all $i \in \mathcal{W}$, we have*

$$\text{CSI}_i(\mathcal{W}) = \text{Sh}_i\left(v^{\mathcal{W}}\right),$$

where $v^{\mathcal{W}}$ is given by (2).

Proof Applying the definitions gives

$$\text{Sh}_i\left(v^{\mathcal{W}}\right)$$
$$= \frac{1}{n!} \sum_{\sigma \in R} \left(\frac{1}{2^{|B_i(\sigma)|-1}} \sum_{T \subseteq B_i(\sigma)} [T \in \mathcal{W}] - \frac{1}{2^{|B_i(\sigma)\setminus\{i\}|-1}} \sum_{T \subseteq B_i(\sigma)\setminus\{i\}} [T \in \mathcal{W}] \right).$$

This can be rearranged as follows:

$$\begin{aligned}\mathrm{Sh}_i\left(v^{\mathcal{W}}\right) &= \frac{1}{n!}\sum_{\sigma\in R}\frac{1}{2^{|B_i(\sigma)|-1}}\sum_{T\subseteq B_i(\sigma):i\in T}\left([T\in\mathcal{W}]+[T\setminus\{i\}\in\mathcal{W}]\right)\\ &\quad-\frac{1}{n!}\sum_{\sigma\in R}\frac{2}{2^{|B_i(\sigma)|-1}}\sum_{T\subseteq B_i(\sigma):i\in T}[T\setminus\{i\}\in\mathcal{W}]\\ &=\frac{1}{n!}\sum_{\sigma\in R}\frac{1}{2^{|B_i(\sigma)|-1}}\sum_{T\subseteq B_i(\sigma):i\in T}\left([T\in\mathcal{W}]-[T\setminus\{i\}\in\mathcal{W}]\right),\end{aligned}$$

which establishes $\mathrm{Sh}\left(v^{\mathcal{W}}\right)=\mathrm{CSI}\left(\mathcal{W}\right)$. □

In light of Theorem 2, we can say that the CSI measures a voter's contribution to the power of the collectivity to act. Moreover, the CSI relates to the PBI as the Shapley value relates to the naïve approach, which assigns to any player the difference between the worth of the grand coalition and its worth after this player has left the game.

This relationship is studied by Casajus and Huettner (2017), who introduce the notion of a decomposer. Beginning from the understanding that $v^{\mathcal{W}}(N)-v^{\mathcal{W}}(N\setminus\{i\})$ is not solely due to i, they are interested in a solution that distributes $v^{\mathcal{W}}(N)-v^{\mathcal{W}}(N\setminus\{i\})$ among all players. It turns out that the natural decomposition is the Shapley value, since the Shapley value is the only decomposition that itself can be decomposed. In the present context, this has the consequence that the CSI is the decomposer of the PBI, with the following implication.[4]

Corollary 1 *For every $\mathcal{W}$ and all $i\in\mathcal{W}$, we have*

$$\mathrm{PBI}_i\left(\mathcal{W}\right)=\mathrm{CSI}_i\left(\mathcal{W}\right)+\sum_{j\in N\setminus\{i\}}\left(\mathrm{CSI}_j\left(\mathcal{W}\right)-\mathrm{CSI}_j\left(\mathcal{W}-i\right)\right), \tag{3}$$

where $\mathcal{W}-i=\{S\in\mathcal{W}\mid i\notin S\}$ represents the committee in which voter j is never supportive of any proposal.

Proof We apply Theorem 2 to the RHS of (3), giving

$$\begin{aligned}&\mathrm{CSI}_i\left(\mathcal{W}\right)+\sum_{j\in N\setminus\{i\}}\left(\mathrm{CSI}_j\left(\mathcal{W}\right)-\mathrm{CSI}_j\left(\mathcal{W}-i\right)\right)\\ &=\mathrm{Sh}_i\left(v^{\mathcal{W}}\right)+\sum_{j\in N\setminus\{i\}}\left(\mathrm{Sh}_j\left(v^{\mathcal{W}}\right)-\mathrm{Sh}_j\left(v^{\mathcal{W}-i}\right)\right)\end{aligned}$$

[4]Equation (3) is not only satisfied by the CSI. Yet, CSI is the only index that itself is decomposable. Indeed, decomposability (i.e., requiring that there exists an index that decomposer) and 2CPCA-Efficiency are characteristic of the CSI.

$$= \sum_{j \in N} \mathrm{Sh}_j \left(v^{\mathcal{W}}\right) - \sum_{j \in N \setminus \{i\}} \mathrm{Sh}_j \left(v^{\mathcal{W}-i}\right)$$
$$= v^{\mathcal{W}} (N) - v^{\mathcal{W}-i} (N)$$

The claim now follows with $v^{\mathcal{W}-i} (N) = v^{\mathcal{W}} (N \setminus \{i\})$ and Proposition 1. □

The term $\mathrm{CSI}_j (\mathcal{W}) - \mathrm{CSI}_j (\mathcal{W} - i)$ reflects the impact of voter i's threats on voters j due to the fact that j's power (measured by the CSI) changes if i refuses to support any proposal. In this sense, the RHS of (3) consists of the power of voter i and the impact of voter i's threats on all other voters. This expression might help identify voters who are critical for keeping the committee agile even though their influence on whether a proposal passes is low, i.e., voters with a high $\mathrm{CSI}_i (\mathcal{W}) / (\mathrm{PBI}_i (\mathcal{W}) - \mathrm{CSI}_i (\mathcal{W}))$ ratio.

Moreover, (3) clarifies once more why summing up the PBI across all players results in a "double count" of power and yields a sum that is cumbersome to interpret. It further gives plausibility to the following property that is characteristic of the PBI.

2Efficiency. For all $\mathcal{W} \in \mathbb{W}$ and voters i, j, we have $\varphi_i (\mathcal{W}) + \varphi_j (\mathcal{W}) = \varphi_{\widehat{ij}}(\mathcal{W}_{\widehat{ij}})$, where $\mathcal{W}_{\widehat{ij}} = \{S \in \mathcal{W} \mid i \notin S \text{ and } j \notin S\} \cup \{(S \setminus \{i, j\}) \cup \{\widehat{ij}\} \in \mathcal{W} \mid i \in S \text{ or } j \in S\}$ represents the committee with voters $(N \setminus \{i, j\}) \cup \{\widehat{ij}\}$ in which i and j merged into one voter $\widehat{ij}$.

Although of mathematical and philosophical interest, this property appears rather counterintuitive, as one would expect the total power of two voters to change if they join forces.[5] In light of (3), however, 2Efficiency merely means that a change in power is counterweighted by a change of the impact on the power of other voters. This becomes more clear when looking at the examples in the next section.

5 Examples

In this section, we consider some committees to exemplify the usability of the new index. In particular, we use the fact that the CSI constitutes the constructive power part of the PBI while the remainder measures the impact of threats.

[5]Note that although 2Efficiency requires invariance of the merging voters' power, it is silent about the other players' power such that the proportions of power might change.

5.5 Dictator

Let $d \in N$ be a dictator, i.e., $\mathcal{W}_1 = \{S \mid d \in S\}$. Then,

$$\mathrm{CSI}_i(\mathcal{W}_1) = \mathrm{PBI}_i(\mathcal{W}_1) = \mathrm{SSI}_i(\mathcal{W}_1) = \begin{cases} 1, & \text{if } i = d \\ 0, & \text{if } i \neq d. \end{cases}$$

The fact that CSI and PBI are equal in this example indicates that threats have no impact and all power is constructive. Indeed, if the dictator blocks every proposal, a voter without power loses no power. Likewise, the power of the dictator is unaffected if another voter "blocks" every proposal.

5.5 One big, two small

Consider a committee where voter 1 needs either 2 or 3 to win, whereas 2 and 3 together lose, $N = \{1, 2, 3\}$ and $\mathcal{W}_{1b2s} = \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. Table 1 contains the resulting power scores.

Table 1: Power scores for $N = \{1, 2, 3\}$ and $\mathcal{W}_{1b2s} = \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$, where voter 3 receives the same as voter 2; the overall score for $\mathrm{CSI}/(\mathrm{PBI} - \mathrm{CSI})$ is the average $7/4 = 1/3 \times 5/4 + 2/3 \times 2$.

	CSI	PBI	CSI/(PBI − CSI)	SSI
Voter 1	$\frac{5}{12}$	$\frac{9}{12}$	$\frac{5}{4}$	$\frac{8}{12}$
Voter 2	$\frac{2}{12}$	$\frac{3}{12}$	2	$\frac{2}{12}$
Overall	$\frac{9}{12}$	$\frac{15}{12}$	$\frac{7}{4}$	1

The previous two examples clarify that the sum of PBI is not an intuitive measure of overall constructive power present in a committee: A dictator is very decisive, whereas reaching a decision is more difficult in the second example, but $\sum_i \mathrm{PBI}_i(\mathcal{W}_1) = 1 < \frac{15}{12} = \sum_i \mathrm{PBI}_i(\mathcal{W}_{1b2s})$. However, if we measure power in terms of CSI and take into account that the threats in $\mathcal{W}_{1b2s}$ are severe, an overall increase of the PBI becomes plausible. If, for example, voter 1 blocks any decision, then voters 2 and 3 become null voters such that the impact of voter 1's threats equals $4/12$—almost as much as her power.

As mentioned in the previous section, a particularity of the PBI is its property 2Efficiency: if two voters merge, then their power sums up as well. If, for instance, voters 1 and 2 merge, we obtain a committee with the two voters $\widehat{12}$ and 3 where now $\widehat{12}$ is a dictator. Clearly, this alliance changes the impact of 1

and 2 on whether a proposal passes. Whereas a coalition between 1 and 2 or 3 was necessary before, $\widehat{12}$ can now easily decide on a proposal. The PBI, therefore, is a doubtful measure of the impact of whether a proposal passes. Understanding the PBI as a sum of power and impact of threats gives plausibility to 2Efficiency: while the constructive power of 1 and 2 increases from $\mathrm{CSI}_1(\mathcal{W}_{1b2s}) + \mathrm{CSI}_2(\mathcal{W}_{1b2s}) = 7/12$ to $\mathrm{CSI}_{\widehat{12}}(\mathcal{W}_1) = 1$, this is balanced by a loss of threats.

5.5 Unanimity committees

Consider a committee where proposals require unanimous approval, $\mathcal{W}_n = \{N\}$. Then, reaching an agreement is more difficult the larger $N$is. This is captured by all well-known indices, e.g.,

$$\mathrm{CSI}_i(\mathcal{W}_n) = \frac{1}{n2^{2-n}}, \quad \mathrm{PBI}_i(\mathcal{W}_n) = \frac{1}{2^{2-n}}, \quad \mathrm{SSI}_i(\mathcal{W}_n) = \frac{1}{n}.$$

The relation of power to impact from threats also decreases,

$$\mathrm{CSI}_i(\mathcal{W}_n)/(\mathrm{PBI}_i(\mathcal{W}_n) - \mathrm{CSI}_i(\mathcal{W}_n)) = 1/(n-1).$$

While every individual player has less impact on whether an agreement is reached, the threats become relatively stronger, since blocking all proposals means that all players lose their constructive power. This trend—a decreasing ratio of power to the impact of threats for increasing committee size—prevails in other committees as well. Note, however, that adding null voters does not affect the outcomes; e.g., for any $T \subseteq N$ we have $\mathrm{CSI}_i(\mathcal{W}_T) = \frac{1}{|T|2^{2-|T|}}$ if $\mathcal{W}_T = \{S \mid T \subseteq S\}$.

5.5 UN Security Council

Let $N = \{1, \ldots, 15\}$ and $\mathcal{W}_{UN} = \{S \mid \{1, \ldots, 5\} \subseteq S$ and $|S| > 9\}$. Table 2 reports the power scores.

The fact that, on average, power is about a quarter of the impact of threats reveals once more that the threat from blocking a proposal is rather convincing. Not surprisingly, this is more pronounced for the powerful “veto powers” than for the non-permanent members.

5.5 Council of the European Union

Let N contain the 28 countries of the EU. A coalition is winning if it represents 65% of the population and 55% of the countries; i.e., $\mathcal{W}_{EU} = \{S \subseteq N \mid \sum_{i \in S} w_i \geq 332,489,700$ and $|S| \geq 16\}$, where w_i is the number of inhabi-

Table 2: Power scores in the UN Security Council; the overall score for $\text{CSI}/(\text{PBI} - \text{CSI})$ is the average $0.25071 = 5/15 \times 0.18516 + 10/15 \times 0.28348$.

	CSI	PBI	$\frac{\text{CSI}}{\text{PBI}-\text{CSI}}$	SSI
Permanent member	0.00809	0.05176	0.18516	0.19627
Non-permanent member	0.00113	0.00513	0.28348	0.00186
Overall	0.05176	0.31006	0.25071	1

tants in country i.[6] Figure 1 contains the SSI and CSI values as well as the $\text{CSI}/(\text{PBI} - \text{CSI})$ ratio. It becomes clear that the differences between powerful and less powerful countries according to the CSI (and the PBI) are smaller than according to the SSI; e.g., $\text{CSI}_{Germany}/\text{CSI}_{Malta} \approx 4.5$ while $\text{SSI}_{Germany}/\text{SSI}_{Malta} \approx 17$. An interpretation for this is that in the EU Council, the impact on the design of proposals is more concentrated than the impact on whether a proposal passes. Further note that the $\text{CSI}/(\text{PBI} - \text{CSI})$ ratio is 1/6 on average and that this ratio is more balanced in the EU than in the UN Security Council.

6 Concluding remarks

We introduce a new power index, the Coleman-Shapley index (CSI), with the purpose of distributing the Coleman power of a collectivity to act (more precisely its normalized, i.e., doubled, version) among the voters by help of the Shapley value. Imposing structural similarities to the most frequently used indices, we single out the CSI as the only index that serves this purpose. The index supports the idea that voters may have different interests in participating in a vote. The most interested voters would then show up before less interested voters, such that the CSI emerges as the expectation of being pivotal in a coalition of the most interested.

The CSI is the decomposer of the Penrose-Banzhaf index (PBI); i.e., it relates to the PBI in the same way that Shapley value relates to the naïve solution (which looks at the difference between the ability to act of all voters and the ability to act of all but this particular voter). Moreover, the CSI sums up to a well-understood entity and allows for cross-committee comparison.

[6]We use the EU population sizes on 01.01.2017 rounded up to the nearest ten (http://ec.europa.eu/eurostat). Detailed numbers can be found in online supplementary materials.

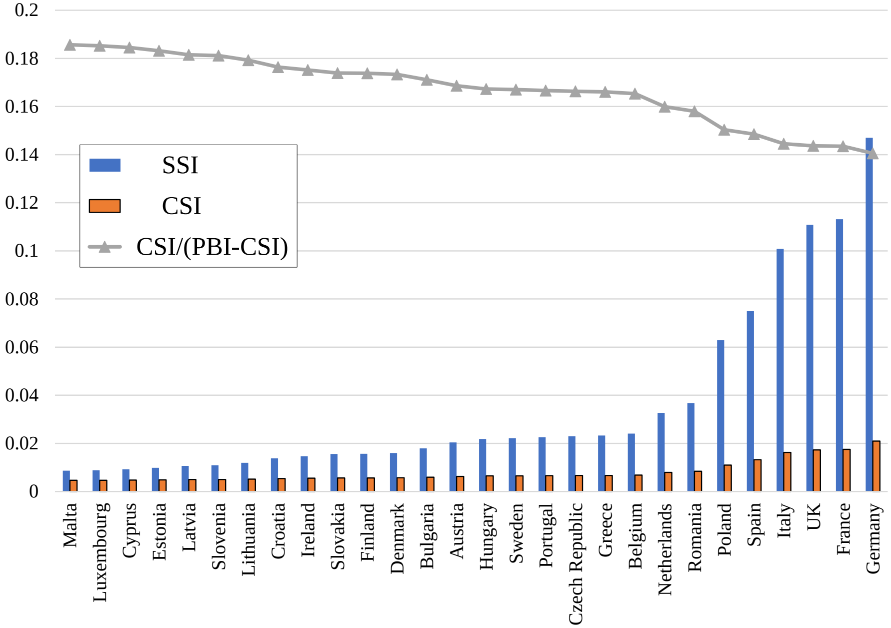

Figure 1: SSI, CSI, and CSI/(PBI-CSI), for the EU Council based on population sizes on 01.01.2017.

This is a requirement when analyzing the consequences of changing majorities (e.g., the consequences of a population-growth-driven change in the set of winning coalitions in the EU council).

Although we advocate the CSI over the PBI for measuring the influence on whether a proposal can pass (I-power in the terminology of Felsenthal and Machover (1998)), using the CSI to measure power reveals another interpretation of the PBI in the sense that it equals the sum of power and the impact of threats. The latter captures the change in power of other voters if a voter no longer supports any proposal. Apparently, the threats tend to become stronger when a committee grows in size.

Because of structural similarities to the Shapley-Shubik index and to the PBI, the CSI serves well as a building block for various extensions. In particular, two-tier voting systems and mutual control structures can be analyzed well based on the CSI. Finally, insights concerning the efficient computation carry over.

A Computation

We first present an effective way to compute the CSI for weighted voting games. Thereafter, we provide an algorithm to recursively compute the CSI via the potential of the Shapley value. Finally, we develop multilinear extension of the CSI that can be used to approximate the CSI for large games along the lines of Leech (2003).

A.A Computation for weighted voting games

Let $\left[q, (w_i)_{i\in N}, k\right]$ denote a weighted voting game with minimal winning coalition size k where coalition S is winning if (i) $\sum_{i\in S} w_i \geq q$ and (ii) $|S| \geq k$. Building upon the study of Kurz (2016) who describes a way to count the number of coalitions of size t that include i and have weight x,

$$c_i\left(x,t\right) = \left|\left\{T \subseteq N \mid i \in T,\ |T| = t, \text{ and } \sum_{j\in T} w_j = x\right\}\right|, \tag{4}$$

in $O(n(1-q+\sum_{j\in N} w_j))$ time and $O(n+1-q+\sum_{j\in N} w_j)$ space, we obtain the CSI as follows:

Proposition 2 *Let $\mathcal{W}$ represent a weighted voting game with quota q, weights $(w_i)_{i\in N}$, and minimal winning coalition size k. Then,*

$$\begin{aligned}\text{CSI}_i\left(\mathcal{W}\right) = &\frac{1}{n!}\sum_{t=k}^{n}(n-t)!\sum_{s=0}^{n-t}\frac{(s+t-1)!}{s!}\frac{1}{2^{s+t-1}}\sum_{x=q}^{q+w_i-1}c_i\left(x,t\right)\\ &+\frac{1}{n!}\left(n-k\right)!\sum_{s=0}^{n-k}\frac{(s+k-1)!}{s!}\frac{1}{2^{s+k-1}}\sum_{x=q+w_i}^{\sum_j w_j}c_i\left(x,k\right).\end{aligned}$$

For $k=1$, the above simplifies to

$$\frac{1}{n!}\sum_{t=1}^{n}(n-t)!\sum_{s=0}^{n-t}\frac{(s+t-1)!}{(s)!}\frac{1}{2^{s+t-1}}\sum_{x=q}^{q+w_i-1}c_i\left(x,t\right).$$

Proof. Denote set of coalitions passing the quota by $\mathcal{W}_1 = \left\{S \mid \sum_{i\in S} w_i \geq q\right\}$, and set of coalitions of size greater k by $\mathcal{W}_2 = \left\{S \mid |S| \geq k\right\}$. We have:

$$\begin{aligned}&\text{CSI}_i\left(\mathcal{W}\right)\\ &= \frac{1}{n!}\sum_{\sigma\in R}\frac{1}{2^{|B_i(\sigma)|-1}}\sum_{T\subseteq B_i(\sigma):i\in T}\left[T\in\mathcal{W} \text{ and } T\setminus\{i\}\notin\mathcal{W}\right]\end{aligned}$$

$$= \frac{1}{n!} \sum_{B \subseteq N : B \ni i} (|B|-1)! \, (n-|B|)! \frac{1}{2^{|B|-1}} \sum_{T \subseteq B : i \in T} [T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}]$$

$$= \frac{1}{n!} \sum_{T \subseteq N : T \ni i} \sum_{S \subseteq N \setminus T} \frac{(s+t-1)! \, (n-s-t)!}{2^{s+t-1}} [T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}]$$

$$= \frac{1}{n!} \sum_{T \subseteq N : T \ni i} \sum_{s=0}^{n-t} \frac{(n-t)!}{s! \, (n-t-s)!} (s+t-1)! \, (n-s-t)! \frac{1}{2^{s+t-1}}$$

$$\times [T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}]$$

$$= \frac{1}{n!} \sum_{t=1}^{n} \sum_{T \subseteq N : T \ni i, |T|=t} (n-t)! \sum_{s=0}^{n-t} \frac{(s+t-1)!}{s! 2^{s+t-1}} [T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}]$$

$$= \frac{1}{n!} \sum_{t=1}^{n} \sum_{T \subseteq N : T \ni i, |T|=t} (n-t)! \sum_{s=0}^{n-t} \frac{(s+t-1)!}{s!} \frac{1}{2^{s+t-1}}$$

$$\times [T \in \mathcal{W}_1, T \setminus \{i\} \notin \mathcal{W}_1 \text{ and } t \geq k]$$

$$+ \frac{1}{n!} \sum_{t=1}^{n} \sum_{T \subseteq N : T \ni i, |T|=t} (n-t)! \sum_{s=0}^{n-t} \frac{(s+t-1)!}{s!} \frac{1}{2^{s+t-1}}$$

$$\times [T \in \mathcal{W}_1, T \setminus \{i\} \in \mathcal{W}_1, \text{ and } t = k]$$

$$\overset{(4)}{=} \frac{1}{n!} \sum_{t=k}^{n} (n-t)! \sum_{s=0}^{n-t} \frac{(s+t-1)!}{s! 2^{s+t-1}} \sum_{x=q}^{q+w_i-1} c_i(x,t) + \frac{(n-k)!}{n!} \sum_{s=0}^{n-k} \frac{(s+k-1)!}{s! 2^{s+k-1}}$$

$$\times \sum_{x=q+w_i}^{\sum_j w_j} c_i(x,k)$$

which completes the proof. □

A.A Computation via Potential of the Shapley value

In order to compute the Shapley value exactly for any TU game, one can use the potential function, recursively defined by $P(\emptyset) = 0$ and $P(S) = v(S) + \sum_{i \in S} P(S \setminus \{i\})$, since $\mathrm{Sh}_i(v) = P(N) - P(N \setminus \{i\})$. In order to compute the CSI, we need to keep track of $v^{\mathcal{W}}(S) = \frac{1}{2^{|S|-1}} \sum_{T \subseteq S} [T \in \mathcal{W}]$ as well. To this end, we need to know about the number of winning subcoalitions of S. This can be done recursively as well. Since $\sum_{i \in S} \sum_{T \subseteq S \setminus \{i\}} v(T) = \sum_{T \subsetneq S} v(T)(s-t)$, we can count the winning coalitions recursively but need to take care of coalition sizes. For every $T \subseteq N$, let $\omega^T \in \mathbb{R}^{|T|}$ denote the vector that contains for each coalition size $k = 1, ..., |T|$ the number of winning coalitions. Then, $\sum_{T \subseteq S : |T| = |S|-1} \frac{\omega_k^T}{|T|-k+1} = \omega_k^S$ for $k = 1, ..., |T|$ and $v^{\mathcal{W}}(S) = \frac{1}{2^{|S|-1}} \sum_k \omega_k^S$.

This motivates the following pseudo code for the iterative computation of CSI.

Input: set of voters $N = \{1, \dots, n\}$, set of winning coalitions $\mathcal{W}$
Output: vector CSI containing $\text{CSI}_i(\mathcal{W})$
$P(\emptyset) \longleftarrow 0$
for every $t = 0, ..., n$ **do**:
    for every $S \subseteq N$ such that $|S| = t + 1$ **do**:
        if $S \in \mathcal{W}$:
            then $\omega_s^S \longleftarrow 1$
            else $\omega_s^S \longleftarrow 0$
    end
    for every $T \subseteq N$ such that $|T| = t$ **do**:
        for every $i \in N \setminus T$ **do**:
            for every $k = 1 \dots t$ **do**:
                $\omega_k^{T\cup\{i\}} \longleftarrow \omega_k^{T\cup\{i\}} + \omega_k^T / (t - k + 1)$
            end
        end
    end
    for every $S \subseteq N$ such that $|S| = t + 1$ **do**:
        $P(S) \longleftarrow \frac{1}{2^{|S|-1}} \sum_{k=1}^{|S|} \omega_k^S$
    end
    for every $T \subseteq N$ such that $|T| = t$ **do**:
        for every $i \in N \setminus T$ **do**:
            $P(T \cup \{i\}) \longleftarrow P(T \cup \{i\}) + \frac{P(T)}{t+1}$
        end
    end
end
for every $i = 1, ..., n$ **do**:
    $\text{CSI}_i(\mathcal{W}) = P(N) - P(N \setminus \{i\})$
end

Clearly, $P(S) = 0$ if $v(T) = 0$ for all T with $|T| \leq |S|$, which allows to exploit knowledge about the minimal winning coalition size. Likewise, one may reduce the tracking of ω_k^S to those k that are greater than the minimal winning coalition size.

A.A Multilinear extension

Fix the player set N and denote the set of all games on N by $\mathbb{V}$. The **multilinear extension** Owen (1972) $\bar{v} : [0,1]^N \to \mathbb{R}$ of a TU game $v \in \mathbb{V}$ is given by

$$\bar{v}(x) = \sum_{S \subseteq N : S \neq \emptyset} v(S) \cdot \prod_{i \in S} x_i \prod_{i \in N \setminus S} (1 - x_i) \qquad \text{for all } x \in [0,1]^N . \tag{5}$$

Owen (1972) shows that the Shapley value can be calculated using the *partial* derivatives of the multi-linear extension as follows: For all v and $i \in N$, we have

$$\text{Sh}_i(v) = \int_0^1 \frac{\partial \bar{v}}{\partial x_i}(\theta, \theta, \ldots, \theta)\, d\theta. \tag{6}$$

That is, the Shapley payoffs are the players' expected marginal productivities along the diagonal of the standard cube representing the players' probabilities with the uniform distribution on the diagonal. For the CSI we find the following expression. For $T \subseteq N, T \neq \emptyset$, the game $u_T \in \mathbb{V}$ given by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called a **unanimity game**. As pointed out in Shapley (1953), these unanimity games form a basis of the vector space[7] $\mathbb{V}$, i.e., any $v \in \mathbb{V}$ can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \tag{7}$$

where the **Harsanyi dividends** $\lambda_T(v)$ can be determined recursively via $\lambda_T(v) = v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v)$ for all $T \subseteq N, T \neq \emptyset$ (see Harsanyi 1959).

The CSI generalizes naturally to a solution for TU games as follows:

$$\text{CSI}_i(v) = \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{2^{t-1} t}$$

for all $v \in \mathbb{V}$ and $i \in N$. Note that $\text{CSI}_i(w) = \text{CSI}_i(\mathcal{W})$ if w is the game representing $\mathcal{W}$, i.e., $w(S) = 1$ if $S \in \mathcal{W}$ and $w(S) = 0$ if $S \notin \mathcal{W}$. The following proposition gives a formula of CSI via the multilinear extension that can be useful for computing the index for large games.

Proposition 3 *For all* $v \in \mathbb{V}$ *and* $i \in N$, *we have*

$$\text{CSI}_i(v) = 2 \int_0^{\frac{1}{2}} \frac{\partial \bar{v}}{\partial x_i}(\theta, \theta, \ldots, \theta)\, d\theta.$$

Proof. Fix $T \subseteq N, T \neq \emptyset$. By (5), we have $\bar{u}_T(x) = \prod_{\ell \in T} x_\ell$ and therefore

$$\frac{\partial \bar{u}_T(x)}{\partial x_i} = \begin{cases} \prod_{\ell \in T \setminus \{i\}} x_\ell, & i \in T, \\ 0, & i \in N \setminus T \end{cases} \qquad \text{for all } x \in [0,1]^N. \tag{8}$$

With $\int_0^{\frac{1}{2}} x^{t-1} dx = \frac{1}{t} x^t \Big|_0^{\frac{1}{2}} = \frac{1}{t}\frac{1}{2^t}$ we get $\text{CSI}_i(\bar{u}) = 2\int_0^{\frac{1}{2}} \frac{\partial \bar{u}_T(x)}{\partial x_i}$. The claim now follows from linearity of the multi-linear extension, of its partial derivatives,

[7]For $v, w \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$, the games $v + w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$.

and of CSI . □

B EU Council data

Country	Population	SSI	CSI	PBI	CSI/(PBI -CSI)
Malta	46030	0.00864677	0.00467474	0.0298542	0.185656881
Luxembourg	59067	0.00882527	0.00470043	0.03007219	0.185262276
Cyprus	85480	0.00919977	0.00475264	0.03051504	0.184479707
Estonia	131564	0.00982827	0.00484335	0.03128485	0.183172286
Latvia	195012	0.01069484	0.0049682	0.0323446	0.181477477
Slovenia	206590	0.0108584	0.00499101	0.03253836	0.181179315
Lithuania	284790	0.01196332	0.00514449	0.03384352	0.179256581
Croatia	415421	0.01379073	0.00539998	0.03601063	0.176408538
Ireland	478438	0.0146639	0.00552318	0.03705747	0.175148386
Slovakia	543534	0.01558604	0.00565048	0.03813914	0.173921608
Finland	550330	0.01567687	0.00566378	0.03825193	0.173798758
Denmark	574877	0.0160319	0.00571161	0.03865863	0.173357408
Bulgaria	710186	0.01794073	0.00597544	0.04090185	0.171086579
Austria	877286	0.02037591	0.00630017	0.04366372	0.168618078
Hungary	979756	0.02183706	0.00649936	0.04535731	0.167259467
Sweden	999515	0.02213365	0.00653773	0.0456836	0.167009444
Portugal	1030957	0.02257792	0.00659883	0.04620332	0.166618229
Czech Republic	1057882	0.02297514	0.00665123	0.0466492	0.166289189
Greece	1076819	0.02324958	0.00668782	0.04696088	0.166061879
Belgium	1135173	0.02409938	0.00680144	0.04792701	0.165382267
Netherlands	1708151	0.03266343	0.00791782	0.05743203	0.159910054
Romania	1964435	0.03673456	0.00842906	0.06176846	0.158026899
Poland	3797296	0.06283548	0.01094856	0.0837624	0.150363722
Spain	4652802	0.0750305	0.01322536	0.10228021	0.148508026
Italy	6058944	0.10084021	0.01623156	0.12854309	0.144522651
UK	6580857	0.11080917	0.01727328	0.13753348	0.143632557
France	6698908	0.11316	0.01753157	0.13973897	0.143457516
Germany	8252165	0.1469712	0.02094077	0.16995624	0.14052749
Overall	51152265	1	0.22657389	1.65293233	0.167156902

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