Blocking cooperation, the equal surplus division value, and auction games

Frank Huettner[*‡] and André Casajus[*†]

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Abstract

We provide new characterizations of the equal surplus division value and the equal division value. This way, the difference between the Shapley value, the equal division value, and the equal surplus division value is pinpointed to one axiom. We use these insights to provide a characterization of the ES-value on auction games. Finally, we obtain a characterization of allocation rules that correspond to Pareto efficient and envy-free allocation-compensation schemes in the context of dividing an indivisible good via auctions with bidder ring formation.

Key words: Solidarity, nullifying player, equal division value, equal surplus division value, envy-freeness

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[*]Professur für Mikroökonomik, Wirtschaftswissenschaftliche Fakultät, Universität Leipzig, Grimmaische Str. 12, 04009 Leipzig, Germany.

[†]LSI Leipziger Spieltheoretisches Institut, Leipzig, Germany.

[‡]corresponding author, mail@frankhuettner.de

1. Introduction

The class of convex mixtures of the equal surplus division value (*ES-value*) and the equal division value (*ED-value*) for cooperative games with transferable utility (*TU-games*) is particularly interesting in the context of the allocation of an indivisible good via auctions. Graham, Marshall and Richard (1990) derive a TU-game from the auction setup, which captures the consequences of bidder rings. The allocations resulting from the application of the Shapley value to this TU-game are not envy-free. van den Brink (2007) shows that the envy-free and Pareto-efficient allocations coincide with the payoffs prescribed by the convex mixtures of the ES-value and the ED-value. Hence, one might be interested in the properties that actually drive these concepts.

van den Brink (2007) suggests a surprising characterization of the ED-value that employs the *nullifying player property* instead of the null player property in the standard characterization of the Shapley value (Shapley, 1953), which also includes efficiency, additivity, and the equal treatment property. A player is called *nullifying* if his presence in a coalition prevents production, i.e., if his presence renders a coalition's worth zero. The nullifying player property requires a nullifying player to obtain a zero payoff. This way, the difference between the Shapley value and the ED-value can be pinpointed to just one axiom. Further characterizations of the ED-value are provided by Driessen and Funaki (1997) and van den Brink and Funaki (2009).

van den Brink (2007) also provides a characterization of the ES-value, which deviates from his characterization of the ED-value above by restricting the nullifying player property to zero-normalized games and adding the invariance property. We feel that this characterization has a minor drawback. For, one would like to have that the difference of the solution concepts manifests itself in a single axiom.

We provide a characterization of the ES-value that replaces the null player property in the standard characterization of the Shapley value with the dummifying player property. A player is called *dummifying* if his presence in a coalition prevents cooperation, so that the players in this coalition fall back to their singleton productivities. The dummifying player property then requires a dummifying player to obtain his singleton worth as payoff. Another two axioms employed by van den Brink (2007) in order to characterize the ED-value are modified in order to characterize the ES-value.

Further, we review characterizations of the ED-value and the Shapley value applied to auction games (van den Brink, 2004). We use our results insights to provide a characterization of the ES-value on auction games. Finally, we obtain a characterization of allocation rules that correspond to Pareto efficient and envy-free allocation-compensation schemes.

This paper is organized as follows. Basic definitions and notations are given in Section 2. Section 3 provides new characterizations of the ES-value. In Section 4, we deal with an application to auction games.

2. Basic definitions and notation

A **(TU-)game** is a pair (N, v) consisting of a non-empty and finite set of players N and a **coalition function** $v \in \mathbb{V}(N) := \left\{f : 2^N \to \mathbb{R} \mid f(\emptyset) = 0\right\}$. Since we work within a fixed player set, we frequently drop the player set as an argument. In particular, we address $v \in \mathbb{V}$ as a game. Subsets of N are called **coalitions**; $v(S)$ is called the worth of coalition S. For $v, w \in \mathbb{V}$, $\lambda \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}$ and $\lambda \cdot v \in \mathbb{V}$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\lambda \cdot v)(S) = \lambda \cdot v(S)$ for all $S \subseteq N$.

For $x \in \mathbb{R}^N$, the **modular game** $m_x \in \mathbb{V}$ is given by $m_x(S) := \sum_{i \in S} x_i$. A game $v \in \mathbb{V}$ is called **zero-normalized** if $v(\{i\}) = 0$ for all $i \in N$; for $v \in \mathbb{V}$, the **associated zero-normalized game** $v^0 \in \mathbb{V}$ is given by $v^0(S) := v(S) - \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. Player $i \in N$ is called a **dummy player** in $v \in \mathbb{V}$ if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subseteq N \setminus i$; player $i \in N$ is called a **null player** in $v \in \mathbb{V}$ if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. Players $i, j \in N$ are called **symmetric** in $v \in \mathbb{V}$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A **TU-value** on N is an function φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $v \in \mathbb{V}$. The **Shapley value** (Shapley, 1953) is given by

$$\mathrm{Sh}_i(v) := \sum_{S \subseteq N : S \ni i} \binom{|N|}{|S|}^{-1} \cdot \frac{1}{|S|} \cdot \left(v(S) - v(S \setminus \{i\})\right), \qquad \text{for all } i \in N.$$

The **equal division value (ED-value)** is given by

$$\mathrm{ED}_i(v) := \frac{v(N)}{|N|}, \qquad \text{for all } i \in N. \tag{1}$$

The **equal surplus division value (ES-value)** (Driessen and Funaki, 1991) is given by

$$\mathrm{ES}_i(v) := v(\{i\}) + \frac{v(N) - \sum_{j\in N} v(\{j\})}{|N|}, \qquad \text{for all } i \in N. \tag{2}$$

Later on, we employ the following standard axioms for TU-values on N.

Efficiency, E. For all $v \in \mathbb{V}$, $\sum_{i\in N} \varphi_i(v) = v(N)$.

Null player, N. For all $v \in \mathbb{V}$ and every $i \in N$, who is a null player in v, $\varphi_i(v) = 0$.

Additivity, A. For all $v, w \in \mathbb{V}$, $\varphi(v + w) = \varphi(v) + \varphi(w)$.

Linearity, A. For all $v, w \in \mathbb{V}$ and $\lambda \in \mathbb{R}$, $\varphi(v + w) = \varphi(v) + \varphi(w)$ and $\varphi(\lambda \cdot v) = \lambda \cdot \varphi(v)$.

Equal treatment, ET. For all $v \in \mathbb{V}$ and $i, j \in N$, who are symmetric in v, $\varphi_i(v) = \varphi_j(v)$.

Symmetry, S. For all $v \in \mathbb{V}$, $i \in N$, and all bijections $\pi : N \to N$, $\varphi_{\pi(i)}(v \circ \pi^{-1}) = \varphi_i(v)$, where $v \circ \pi^{-1} \in \mathbb{V}$ is given by $v \circ \pi^{-1}(S) = v(\pi^{-1}(S))$, $S \subseteq N$.

3. Dummifying players

The standard characterization of the Shapley value (Shapley, 1953) employs efficiency, additivity, the equal treatment property, and the null player property. The null player property indicates the fact that the Shapley value particularly reflects a player's own productivity. Within the null player property, van den Brink (2007) replaces null players by nullifying players. Player $i \in N$ is **nullifying** in $v \in \mathbb{V}$ if $v(S) = 0$ for all $S \subseteq N$ such that $i \in S$. This yields the following axiom.

Nullifying player, Ng. For all $v \in \mathbb{V}$ and $i \in N$ such that i is nullifying in v, we have $\varphi_i(v) = 0$.

According to the nullifying player property, a zero payoff is assigned to nullifying players, i.e., to players whose presence renders a coalition's worth zero. Replacing the null player property by the nullifying player property, he obtains a new characterization of the ED-value.

Theorem 1 (van den Brink 2007)**.** *The ED-value is the unique TU-value that satisfies efficiency (**E**), additivity (**A**), the equal treatment property (**ET**), and the nullifying player property (**Ng**).*

van den Brink (2007, Theorem 4.1) also provides a characterization of the ES-value, which deviates from his characterization of the ED-value above by restricting the nullifying player property to zero-normalized games and adding the invariance property. However, we feel that making use of the restricted nullifying property somewhat blurs the characteristic property of the equal surplus division rule.

A nullifying player does not only obstruct cooperation within any coalition containing him, but also neutralizes the productive potential of such a coalition. Dropping the latter feature of a nullifying player leads to the notion of a dummifying player, i.e., a player whose presence rules out any cooperation but does not neutralize the stand-alone productivities of the players in his coalition. Formally, a player $i \in N$ is **dummifying** in $v \in \mathbb{V}$ if $v(S) = \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$ such that $i \in S$. Analogously to the nullifying player property one obtains the dummifying player property below.

Dummifying player, Dg. For all $v \in \mathbb{V}$ and $i \in N$ such that i is dummifying in v, we have $\varphi_i(v) = v(\{i\})$.

According to the dummifying player property, the singleton worth is assigned to dummifying players. Replacing the null player property in the standard characterization of the Shapley value by the dummifying player property, one obtains a characterization of the ES-value.[1]

Theorem 2. *The ES-value is the unique TU-value that satisfies efficiency (**E**), additivity (**A**), the equal treatment property (**ET**), and the dummifying player property (**Dg**).*

Proof. By (2), it is clear that ES obeys **E**, **A**, and **ET**. Let $i \in N$ be dummifying in $v \in \mathbb{V}$. Hence, $v(N) = \sum_{j \in N} v(\{j\})$. By (2), we have $\mathrm{ES}_i(v) = v(\{i\})$. Thus, ES meets **Dg**.

Now, let φ be a TU-value on N that satisfies **E**, **A**, **ET**, and **Dg**. Clearly, **Dg** implies the restriction of **Ng** to zero-normalized games. In view of (the proof of) van den Brink (2007, Theorem 3.1), it suffices to show that **A** and **Dg** imply $\varphi(v + m_b) = \varphi(v) + b$ for all $v \in \mathbb{V}$ and $b \in \mathbb{R}^N$. By **A**, we have $\varphi(v + m_b) = \varphi(v) + \varphi(m_b)$. Since all players are dummifying in m_b, **Dg** implies $\varphi(m_b) = b$ and we are done. □

[1] One easily checks that all characterizations in this paper are non-redundant.

van den Brink (2007) suggests another two axioms in order to characterize the ED-value. The first axiom, coalitional standard equivalence[2] is related to a version of Chun's (1989) coalitional strategic equivalence[3]. The second one, coalitional monotonicity[4], is related to Young's (1985) strong monotonicity[5]. In coalitional standard equivalence, we replace nullifying players by dummifying players and obtain coalitional surplus equivalence below. In order to accommodate to the ES-value, we further modify coalitional monotonicity yielding coalitional surplus monotonicity below.

Coalitional surplus equivalence, CSE. For all $v, w \in \mathbb{V}$ and $i \in N$ such that i is dummifying in w, we have $\varphi_i(v+w) = \varphi_i(v) + w(\{i\})$.

Coalitional surplus monotonicity, CSM. For all $v, w \in \mathbb{V}$ and $i \in N$ such that $v^0(S) \geq w^0(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(v) - v(\{i\}) \geq \varphi_i(w) - w(\{i\})$.

Replacing coalitional standard equivalence in van den Brink's (2007) characterization of the ED-value by coalitional surplus equivalence, one obtains a characterization of the ES-value.

Theorem 3. *The ES-value is the unique TU-value that satisfies efficiency (**E**), the equal treatment property (**ET**), and either (i) coalitional surplus equivalence (**CSE**) or (ii) coalitional surplus monotonicity (**CSM**).*

Proof. By (2), it is clear that ES obeys **E**, **ET**, **CSE**, and **CSM**. Now, let φ be a TU-value on N that satisfies **E**, **ET**, and **CSE**.

For all $v \in \mathbb{V}$, all players are dummifying in $v - v^0$. By **CSE**, $\varphi_i(v) = \varphi_i(v^0) + v(\{i\})$ for all $i \in N$. Therefore, it suffices to restrict attention to the class of zero-normalized games. For this class, **CSE** becomes coalitional standard equivalence. Moreover, the proof of van den Brink (2007, Theorem 3.2) works within the class of zero-normalized games. Hence, $\varphi = \text{ES}$ for zero-normalized games, what establishes uniqueness for (i). Part (ii) follows from the observation that **CSM** implies **CSE**. □

[2]A solution φ satisfies coalitional standard equivalence if for all $v, w \in \mathbb{V}$ and $i \in N$ such that i is a nullifying player in w, we have $\varphi_i(v+w) = \varphi_i(v)$.

[3]A solution φ satisfies coalitional strategic equivalence if for all $v, w \in \mathbb{V}$ and $i \in N$ such that i is a null player in w, we have $\varphi_i(v+w) = \varphi_i(v)$.

[4]A solution φ satisfies coalitional monotonicity if for all $v, w \in \mathbb{V}$ and $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(v) \geq \varphi_i(w)$.

[5]A solution φ satisfies strong monotonicity if for all $v, w \in \mathbb{V}$ and $i \in N$ such that $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v) \geq \varphi_i(w)$.

In analogy to van den Brink (2007, Theorem 3.3), we establish that the equal treatment property can be relaxed into the weak equal treatment property in the second part of the above theorem.

Weak equal treatment, ET$^-$. For all $v \in \mathbb{V}$ such that all players are pairwise symmetric in v, we have $\varphi_i(v) = \varphi_j(v)$ for all $i, j \in N$.

Theorem 4. *The ES-value is the unique TU-value that satisfies efficiency (**E**), the weak equal treatment property (**ET$^-$**), and coalitional surplus monotonicity (**CSM**).*

Proof. By (2), it is clear that ES obeys **E**, **ET$^-$**, and **CSM**. Now, let φ be a TU-value on N that satisfies **E**, **ET$^-$**, and **CSM**.

We adapt the proof of van den Brink (2007, Theorem 3.3). For $v \in N$, define $w \in \mathbb{V}$ by

$$w(S) = \begin{cases} v^0(N), & S = N, \\ \min_{T \subseteq N} v^0(T), & S \subsetneq N, \ |S| > 1, \\ 0, & S \subsetneq N, \ |S| \leq 1. \end{cases}$$

By **E** and **ET$^-$**, $\varphi_j(w) = v^0(N) / |N|$ for all $j \in N$. Since $v^0(S) \geq w^0(S)$ for all $S \subseteq N$, **CSM** implies

$$\varphi_i(v) - v(\{i\}) \geq \varphi_i(w) - w(\{i\}) = v^0(N) / |N|$$

for arbitrary $i \in N$. Hence, $\varphi_i(v) \geq \mathrm{ES}_i(v)$ for all $i \in N$. By **E**, we have $\varphi(v) = \mathrm{ES}(v)$. □

The theorems in this section show that the equal surplus division rule treats dummifying players as the Shapley value treats dummy players, while the equal division rule handles nullifying players as the Shapley value treats null players.

4. Auction games

In this section, we consider a situation where an indivisible good has to be allocated to one of n agents. Each agent $i \in N = \{1, \ldots n\}$ has a non-negative valuation of the good denoted by V_i. Keeping N fixed, such a situation is captured by what we call an **allocation situation**, i.e., a vector of valuations $V \in \mathbb{R}^n_+$. W.l.o.g., we assume $V_i \leq V_j$ for all $i, j \in N$ such that $i < j$. Let $\mathbb{R}^n_{\leq}$ denote the set of such allocation situations.

The agent who obtains the good compensates the other agents by paying them some amount of money. An **allocation-compensation scheme** for an allocation situation is a pair $(i, c) \in N \times \mathbb{R}^n$ where $i \in N$ is the agent who obtains the indivisible good and $c \in \mathbb{R}^n$ is the compensation vector that obeys $\sum_{j \in N} c_j = 0$; c_i denotes the total amount compensation paid by agent i, whereas for $j \neq i$, c_j denotes the compensation paid by agent i to agent j. An allocation-compensation scheme (i, c) induces the payoff vector

$$\pi(i, c) = (c_1, \ldots, V_i + c_i, \ldots, c_n).$$

An allocation compensation scheme (i, c) is called **Pareto efficient** if an agent with the highest valuation obtains the good, i.e., $V_i = V_n$; it is called **envy-free** if no agent favors to be in the shoes of another one, i.e., (i) $V_i + c_i \geq c_j$ for all $j \in N$, (ii) $c_j \geq c_k$ for all $j, k \in N \setminus \{i\}$, and (iii) $c_j \geq V_j + c_i$ for all $j \in N \setminus \{i\}$.

Given an allocation situation, one might be interested in "fair" allocation-compensation schemes. Abstracting from the actual assignment of the indivisible good, one may look for an **allocation rule** $a : \mathbb{R}^n_{\leq} \to \mathbb{R}^n$, i.e., a mapping that assigns to any allocation situation V a payoff vector $a(V)$.

In order to derive a "fair" allocations, one could make use of an auction. Graham et al. (1990) derive the following TU-game from the auction setup, which captures the consequences of bidder coalitions. For all $V \in \mathbb{R}^n_{\leq}$, define $v_V \in \mathbb{V}$ by

$$v_V(S) = \max\left\{\max_{i \in S} V_i - \max_{i \in N \setminus S} V_i, 0\right\} \qquad \text{for all } S \subseteq N. \tag{3}$$

Using this TU-game, any TU-value φ induces an allocation rule a^{φ} given by $a(V) = \varphi(v_V)$ for all $V \in \mathbb{R}^n_{\leq}$. Applying the Shapley value to v_V yields the Shapley allocation rule,

$$a_i^{\mathrm{Sh}}(V) = \mathrm{Sh}_i(v_V) = \sum_{j=1}^{i} \frac{V_j - V_{j-1}}{n - j + 1} \qquad \text{for all } i \in N \text{ and } V \in \mathbb{R}^n_{\leq}.$$

Using the ED-value and the ES-value, respectively, gives the corresponding allocation rules,

$$a_i^{\mathrm{ED}}(V) = \mathrm{ED}_i(v_V) = \frac{V_n}{n}$$

and

$$a_i^{\mathrm{ES}}(V) = \mathrm{ES}_i(v_V) = \begin{cases} V_n - V_{n-1} + \frac{V_{n-1}}{n}, & i = n, \\ \frac{V_{n-1}}{n}, & i = 1, \ldots, n-1. \end{cases}$$

for all $i \in N$ and $V \in \mathbb{R}^n_{\leq}$.

The Shapley allocation rule corresponds to Pareto efficient allocation-compensation schemes, i.e., it every allocation-compensation schemes (i, π) for some $V \in \mathbb{R}^n_{\leq}$

such that $\pi(i,c) = a^{\mathrm{Sh}}(V)$ must be Pareto efficient. However, allocation-compensation schemes with payoffs equal to the Shapley allcoation rule need not necessarly be envy-free. In contrast, both the ED-rule and the ES-rule correspond to allocation-compensation schemes that are Pareto efficient and envy-free. Indeed, van den Brink (2007, Proposition 5.1) shows that exactly those allocation-compensation schemes satisfy both properties if and only if their payoffs equal convex mixtures of the ED-rule payoffs and the ES-rule payoffs.

Proposition 5 (van den Brink 2007)**.** *An allocation-compensation scheme* (i,c) *for* $V \in \mathbb{R}^n_{\leq}$ *is Pareto efficient and envy-free if and only if there is some* $\lambda \in [0,1]$ *such that* $\pi(i,c) = \lambda \cdot a^{\mathrm{ED}}(V) + (1-\lambda) \cdot a^{\mathrm{ES}}(V)$*.*

van den Brink (2004, Theorems 4.2 and 4.4) suggests characterizations for the Shapley allocation rule and of the equal division allocation rule using the following axioms.[6]

Efficiency, E. For all $V \in \mathbb{R}^n_{\leq}$, $\sum_{i\in N} a_i(V) = V_n$.

Symmetry, S. For all $V \in \mathbb{R}^n_{\leq}$ and $i,j \in N$, $V_i = V_j$ implies $a_i(V) = a_j(V)$.

Independence on higher valuations, IHV. For all $V,W \in \mathbb{R}^n_{\leq}$ and $i,j \in N$ such that $i \neq j$, $\min\{W_j, V_j\} \geq V_i$ and $W_h = V_h$ for all $h \in N \setminus \{j\}$, we have $a_i(V) = a_i(W)$.

Independence on lower valuations, ILV. For all $V,W \in \mathbb{R}^n_{\leq}$ and $i,j \in N$ such that $i \neq j$, $\max\{W_j, V_j\} \leq V_i$ and $W_h = V_h$ for all $h \in N \setminus \{j\}$, we have $a_i(V) = a_i(W)$.

Efficiency guarantees that a player with a highest valuation obtains the indivisible good. Symmetry demands that two players with identical valuations are treated in the same way. Independence on higher valuation states that the outcome of some player i remains unchanged if we increase the value of some player that valuates the good at least as high as player i. This axiom is directly obtained by applying Young's strong monotonicity to (3). Independence on lower valuation demands that the outcome of some player i remains unchanged if we decrease the value of some player that valuates the good at most as high as player i.

[6] Abusing notation, "efficiency" and "symmetry" refer both to properties for TU-values and allocation rules.

Theorem 6 (van den Brink 2004). *(i) An allocation rule $a : \mathbb{R}^n_{\leq} \to \mathbb{R}^n$ on auction games is the Shapley allocation rule a^{Sh} if and only if satisfies efficiency (**E**), symmetry (**S**), and independence on higher valuations (**IHV**).*

*(ii) An allocation rule $a : \mathbb{R}^n_{\leq} \to \mathbb{R}^n$ on auction games is the equal division allocation rule a^{ED} if and only if satisfies efficiency (**E**), symmetry (**S**), and independence on lower valuations (**ILV**).*

In the previous section, we employed coalitional surplus equivalence in order to characterize the ES-value. We now explore its meaning in the context of allocation situations. One easily checks that exists a dummifying player in v_Z if and only if and $Z_j = 0$ for all $j \in N \setminus \{n\}$. Indeed, all players are dummifying in such a game. Moreover, we have $v_{V+Z} = v_V + v_Z$ for all $V, Z \in \mathbb{R}^n_{\leq}$. Thus, adding a game with dummifying players corresponds to an increase in the highest valuation. In this case, coalitional surplus equivalence implies that the payoff of the player with the second highest valuation does not change. This corresponds to the following axiom for allocation rules.[7] Note that this axiom is weaker than independence on higher valuations.

Weak independence on higher valuations, IHV$^-$. For all $V, W \in \mathbb{R}^n_{\leq}$ such that $W_h = V_h$ for all $h \in N \setminus \{n\}$, we have $a_{n-1}(V) = a_{n-1}(W)$.

Proposition 7. *If the TU-value φ satisfies coalitional surplus equivalence (**CSE**), then the induced allocation rule a^{φ} satisfies weak independence on higher valuations (**IHV$^-$**).*

While the ES-rule does not obey independence on higher valuations, the previous proposition leads to its weaker cousin above. By Theorem 6 (ii), the ES-rule further fails independence on lower valuations. While weak independence on higher valuations restricts attention to player $n-1$, the ES-rule satisfies a weaker version of independence on lower valuations that disregards this player.

Weak independence on lower valuations, ILV$^-$. For all $V, W \in \mathbb{R}^n_{\leq}$ and $i \in N$ and $j \in N \setminus \{n-1\}$ such that $i \neq j$, $\max\{W_j, V_j\} \leq V_i$ and $W_h = V_h$ for all $h \in N \setminus \{j\}$, we have $a_i(V) = a_i(W)$.

The difference between independence on lower valuation and this axiom is that the latter remains silent for changes of the valuation of player $n-1$. This respects the

[7] In fact, the implications of **CSE** on induced allocation rules are stronger in the sense that a change in V_n fully accrues to player n, while all other players' payoffs remain unaffected.

special meaning of the player with the second highest valuation in auctions games. Indeed, we have the following characterization of the ES-rule.

Theorem 8. *An allocation rule $a : \mathbb{R}^n_{\leq} \to \mathbb{R}^n$ on auction games is the equal surplus division allocation rule a^{ES} if and only if it satisfies efficiency (**E**), symmetry (**S**), weak independence on lower valuations (**ILV**$^-$), and weak independence on higher valuations (**IHV**$^-$).*

Proof. One easily checks that a^{ES} meets the properties of the theorem. Let now the allocation rule a be as the theorem. For $|N| = 1$, **E** already entails $a = a^{\mathrm{ES}}$. Let now $|N| > 1$.

Claim 1: $a_{n-1}(V) = \frac{V_{n-1}}{n}$ for all $V \in \mathbb{R}^n_{\leq}$. This can be see as follows. For $V \in \mathbb{R}^n_{\leq}$, we have

$$\begin{aligned} a_{n-1}(V) &\overset{\mathbf{IHV}^-}{=} a_{n-1}(V_1, \ldots, V_{n-2}, V_{n-1}, V_{n-1}) \\ &\overset{\mathbf{ILV}^-}{=} a_{n-1}(V_{n-1}, \ldots, V_{n-1}) \\ &\overset{\mathbf{E,S}}{=} \frac{V_{n-1}}{n}. \end{aligned}$$

Claim 2: $a_n(V) = \frac{V_{n-1}}{n} + V_n - V_{n-1}$ for all $V \in \mathbb{R}^n_{\leq}$. This can be see as follows. For $V \in \mathbb{R}^n_{\leq}$, we have

$$\begin{aligned} a_n(V) &\overset{\mathbf{ILV}^-}{=} a_n(V_{n-1}, \ldots, V_{n-1}, V_n) \\ &\overset{\mathbf{E},\textit{Claim 1},\mathbf{S}}{=} V_n - (n-1) \cdot \frac{V_{n-1}}{n} = \frac{V_{n-1}}{n} + V_n - V_{n-1}. \end{aligned}$$

Claim 3: $a = a^{\mathrm{ES}}$ for all $V \in \mathbb{R}^n_{\leq}$. Suppose $a(W) \neq a^{\mathrm{ES}}(W)$ for some $W \in \mathbb{R}^n_+$. There is some minimal $k \in N$ such that $a_j(V) = a^{\mathrm{ES}}_j(V)$ for all $j > k$ and all $V \in \mathbb{R}^n_{\leq}$. W.l.o.g., $a_k(W) \neq a^{\mathrm{ES}}_k(W)$. By *Claims 1* and *2*and **S**, $k < n-1$. Consider $W^k := (W_k, \ldots, W_k, W_{k+1}, W_{k+2}, \ldots, W_{n-1}, W_n)$. For all $j > k$, we have $a_j(W^k) = a^{\mathrm{ES}}_j(W^k)$ and therefore

$$\begin{aligned} a_k\left(W^k\right) &\overset{\mathbf{E,S}}{=} \frac{W_n - \sum_{j=k+1}^n a^{\mathrm{ES}}_j\left(W^k\right)}{k} \\ &= \frac{W_n - \left(\frac{W_{n-1}}{n} + W_n - W_{n-1}\right) - ((n-1)-k) \cdot \frac{W_{n-1}}{n}}{k} \\ &= \frac{W_{n-1}}{n} = a^{\mathrm{ES}}_k\left(W^k\right). \end{aligned}$$

Applying **ILV**$^-$ $k-1$ times yields $a_k(W) = a_k\left(W^k\right) = a^{\mathrm{ES}}_k(W)$, a contradiction. $\square$

The previous theorem and Theorem 6 (ii) are particularly interesting in view of Proposition 5. They characterize the payoffs that correspond to the extremal Pareto efficient and envy-free allocation compensation schemes. One may now be interested in a characterization of their mixtures. To this end, note that, besides efficiency and symmetry, both values satisfy weak independence on lower valuations. Apart from this, we just need to strengthen symmetry into desirability (Maschler and Peleg, 1966), and to further weaken weak independence on higher valuations into harmless higher improvement, below.

Desirability, D. For all $V \in \mathbb{R}^n_{\leq}$ and $i, j \in N$, $V_i \geq V_j$ implies $a_i(V) \geq a_j(V)$.

Harmless higher improvement, HHI. For all $V, W \in \mathbb{R}^n_{\leq}$ such that $W_h = V_h$ for all $h \in N \setminus \{n\}$, and $W_n \geq V_n$, we have $a_{n-1}(V) \geq a_{n-1}(W)$.

Desirability rules out that a higher valuation lead to lower payoffs. Harmless higher improvement prevents that an increase in the highest valuation entails a decrease of the payoff of the player with the second highest valuation. As the following theorem shows, these properties together with weak independence on lower valuations correspond to envy-freeness given Pareto efficiency.

Theorem 9. *(i) If for some $V \in \mathbb{R}^n_{\leq}$ the allocation compensation-schemes $\pi(i, c)$ is Pareto efficient and envy-free, then there is an allocation rule $a : \mathbb{R}^n_{\leq} \to \mathbb{R}^n$ satisfying efficiency (**E**), desirability (**D**), weak independence on lower valuations (**ILV**$^-$), and harmless higher improvement (**HHI**), such that $\pi(i, c) = a(V)$.*

*(ii) If the allocation rule $a : \mathbb{R}^n_{\leq} \to \mathbb{R}^n$ satisfies efficiency (**E**), desirability (**D**), weak independence on lower valuations (**ILV**$^-$), and harmless higher improvement (**HHI**), then for all $V \in \mathbb{R}^n_{\leq}$, the allocation compensation-schemes that satisfy $\pi(i, c) = a(V)$ are Pareto efficient and envy-free.*

Remark 1. *According to Proposition 5, there always exists an Pareto efficient and envy-free allocation compensation-schemes. Hence, the statement will never be empty.*

Proof. (i) We show that the allocation rules defined by $a^\lambda := \lambda \cdot a^{\mathrm{ED}} + (1 - \lambda) \cdot a^{\mathrm{ES}}$, $\lambda \in [0, 1]$ satisfy **E**, **D**, **ILV**$^-$, and **HHI**. It is clear that a^{ED} satisfies **HHI**. Since **HHI** is implied by **IHV**$^-$, Theorem 8 implies that a^{ES} also satisfies **HHI**. Now it is obvious that any a^λ inherits **E, D, HHI**, and **ILV**$^-$ from a^{ED} and a^{ES}. The assertion now follows from Proposition 5.

(ii) Let the allocation rule a be as in (ii). For $|N| = 1$, **E** already entails $a = a^{\text{ED}} = a^{\text{ES}}$. Let now $|N| > 1$.

Claim 1: $\frac{V_{n-1}}{n} \leq a_{n-1}(V) \leq \frac{V_n}{n}$ for all $V \in \mathbb{R}^n_{\leq}$. This can be seen as follows. By **ILV**$^-$, $a_{n-1}(V) = a_{n-1}(V_{n-1}, \ldots, V_{n-1}, V_n)$ for all $V \in \mathbb{R}^n_{\leq}$. Therefore,

$$\begin{aligned} \frac{V_{n-1}}{n} &\overset{\mathbf{E,D}}{=} a_{n-1}(V_{n-1}, \ldots, V_{n-1}) \\ &\overset{\mathbf{HHI}}{\leq} a_{n-1}(V_{n-1}, \ldots, V_{n-1}, V_n) \end{aligned}$$

establishes the first inequality. By **E** and **D**, we have $a_{n-1}(V_{n-1}, \ldots, V_{n-1}, V_n) \leq \frac{V_n}{n}$, implying the second inequality. We may therefore define

$$\lambda_V := \begin{cases} \frac{n \cdot a_{n-1}(V) - V_{n-1}}{V_n - V_{n-1}}, & V_n > V_{n-1}, \\ 0, & V_n = V_{n-1}, \end{cases} \qquad \text{for } V \in \mathbb{R}^n_{\leq}. \tag{4}$$

Claim 2: $a_n(V) = \lambda_V \cdot a_n^{\text{ED}}(V) + (1 - \lambda_V) \cdot a_n^{\text{ES}}(V)$ for all $V \in \mathbb{R}^n_{\leq}$ with λ_V given by (4). This can be seen as follows. For all $V \in \mathbb{R}^n_{\leq}$, we have

$$\begin{aligned} a_n(V) &\overset{\mathbf{ILV}^-}{=} a_n(V_{n-1}, \ldots, V_{n-1}, V_n) \\ &\overset{\mathbf{E,D}}{=} V_n - (n-1) \cdot a_{n-1}(V_{n-1}, \ldots, V_{n-1}, V_n) \\ &\overset{(4)}{=} V_n - (n-1) \cdot \frac{\lambda_V \cdot (V_n - V_{n-1}) + V_{n-1}}{n} \\ &= \lambda_V \cdot \frac{V_n}{n} + (1 - \lambda_V) \cdot \left(\frac{V_{n-1}}{n} + V_n - V_{n-1} \right), \end{aligned}$$

confirming the claim.

Claim 3: $a(V) = \lambda_V \cdot a^{\text{ED}}(V) + (1 - \lambda_V) \cdot a^{\text{ES}}(V)$ for any $V \in \mathbb{R}^n_{\leq}$ with λ_V given by (4). Suppose this is not true for some $W \in \mathbb{R}^n_{+}$. Then there is some minimal $k \in N$ such that $a_j(V) = \lambda_V \cdot a_j^{\text{ED}}(V) + (1 - \lambda_V) \cdot a_j^{\text{ES}}(V)$ for all $j > k$ and all $V \in \mathbb{R}^n_{\leq}$. W.l.o.g., $a_k(W) \neq \lambda_W \cdot a_k^{\text{ED}}(W) + (1 - \lambda_W) \cdot a_k^{\text{ES}}(W)$. By *Claims 1* and *2* and **S**, $k < n - 1$. Consider $W^k := (W_k, \ldots, W_k, W_{k+1}, W_{k+2}, \ldots, W_{n-1}, W_n)$. For all $j > k$, we have $a_j\left(W^k\right) = \lambda_{W^k} \cdot a_j^{\text{ED}}\left(W^k\right) + (1 - \lambda_{W^k}) \cdot a_j^{\text{ES}}\left(W^k\right)$ and therefore

$$\begin{aligned} a_k\left(W^k\right) &\overset{\mathbf{E,S}}{=} \frac{W_n - \sum_{j=k+1}^{n} \left(\lambda_{W^k} \cdot a_j^{\text{ED}}\left(W^k\right) + (1 - \lambda_{W^k}) \cdot a_j^{\text{ES}}\left(W^k\right)\right)}{k} \\ &= \frac{\lambda_V \cdot (W_n - W_{n-1}) + W_{n-1}}{n} \\ &= \lambda_{W^k} \cdot a_k^{\text{ED}}\left(W^k\right) + (1 - \lambda_{W^k}) \cdot a_k^{\text{ES}}\left(W^k\right). \end{aligned}$$

Applying **ILV**$^-$ $k - 1$ times yields $a_k(W) = a_k\left(W^k\right) = \lambda_{W^k} \cdot a_k^{\text{ED}}\left(W^k\right) + (1 - \lambda_{W^k}) \cdot a_k^{\text{ES}}\left(W^k\right)$, a contradiction.

The assertion now follows from Proposition 5. □

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